CLOSED MANIFOLDS COMING FROM ARTINIAN COMPLETE INTERSECTIONS

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ABSTRACT. We reformulate the integrality property of the Poincaré inner product in the middle dimension, for an arbitrary Poincaré $\mathbb{Q}$-algebra, in classical terms (discriminant and local invariants). When the algebra is 1-connected, we show that this property is the only obstruction to realizing it by a closed manifold, up to dimension 11. We reinterpret a result of Eisenbud and Levine on finite map germs, relating the degree of the map germ to the signature of the associated local ring, to answer a question of Halperin on artinian weighted complete intersections. We analyse the homogeneous artinian complete intersections over $\mathbb{Q}$ realized by closed manifolds of dimensions 4 and 8, and their signatures.

1. INTRODUCTION

1.1. **Artinian complete intersection.** Let $\mathcal{A}$ be a *weighted artinian complete intersection* ($WACI$), that is, a commutative graded $\mathbb{Q}$-algebra of the form

$$\mathcal{A} = \mathbb{Q}[x_1, \ldots, x_n]/\mathcal{I}\,, \tag{1.1}$$

where the variables x_i have positive even weights, $w_i :=| x_i |$, and the ideal $\mathcal{I}$ is generated by a regular sequence,

$$\mathcal{I} = (f_1, \ldots, f_n)\,, \tag{1.2}$$

of weighted-homogeneous polynomials, f_i.

One knows [5, Theorem 3 and p.198] that $\mathcal{A}^*$ is a 1-connected *rational Poincaré duality algebra* ($\mathbb{Q}$–PDA), with Poincaré polynomial

$$\mathcal{A}^*(t) = \prod_{i=1}^{n} \frac{1 - t^{|f_i|}}{1 - t^{|x_i|}} \tag{1.3}$$

(and, consequently, with even formal dimension, $m = \sum_{i=1}^{n}(| f_i | - | x_i |)$).

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1.2. **The integrality obstruction.** Let $\mathcal{A}^*$ be an arbitrary 1-connected Poincaré duality $\mathbb{Q}$-algebra, with formal dimension m. The *smoothing problem* we are going to look at is the following:

> is $\mathcal{A}^*$ isomorphic to a graded algebra of the form $H^*(M^m, \mathbb{Q})$, where M is a 1-connected closed smooth m-manifold?

We shall say that $\mathcal{A}$ is *smoothable* if the answer is yes.

By $\mathbb{Q}$-surgery ([12], [2]), we know that $\mathcal{A}$ is smoothable, for $m \neq 4k$. Assume now that $m = 4k$, and pick an *orientation*, $\omega \in \mathcal{A}^{4k} \setminus \{0\}$. This gives rise (via Poincaré duality) to a symmetric inner product space over $\mathbb{Q}$, denoted by $(\mathcal{A}^{2k}, \cdot_\omega) \in W(\mathbb{Q})$. (Here and in the sequel, $W(R)$ denotes the Witt group of the ring R; see [8].) If $\mathcal{A}$ is smoothable, then clearly

$$(\mathcal{A}^{2k}, \cdot_\omega) \in W(\mathbb{Z}), \quad \text{for some orientation} \quad \omega\,. \tag{1.4}$$

It turns out that the *integrality obstruction* from (1.4), for a fixed orientation ω, is equivalent to the fact that the quadratic form on $\mathcal{A}^{2k}$ associated to $\cdot_\omega$ is a sum of signed squares, over $\mathbb{Q}$; see [8, Corollary IV.2.6].

When the signature is zero, the integrality condition is equivalent to $(\mathcal{A}^{2k}, \cdot)$ being split; see [8, I.6–7]. In this case, (1.4) is the only obstruction to smoothing; see [12] and [2], and also [10, Proposition 3.4]. In the non-zero signature case, additional obstructions may appear, see e.g. [10, §4.5] for some simple examples, based on [2] and [3].

1.3. **Main results.** The smoothing problem described above may be solved by using fundamental $\mathbb{Q}$–surgery results due to D. Sullivan; see [12], [2]. This opens the way for constructing closed manifolds with interesting geometric properties, starting from $\mathbb{Q}$–PDA's (see [10] for applications to geodesics).

The difficulties of the smoothing problem stem from the fact that the obstructions involve, besides (1.4), delicate conditions on the signature. If $\mathcal{A}$ is an arbitrary $WACI$, we point out a topological interpretation of the signature, in terms of the defining relations of $\mathcal{A}$, thus answering a question of S. Halperin from [5]. We do this in Section 2, by using a basic result on finite map germs, due to D. Eisenbud and H. Levine [4].

In Section 3, we focus on the integrality obstruction. In Theorem 3.2, we show that (1.4) is the only obstruction to smoothing, for arbitrary $\mathbb{Q}$–PDA's of formal dimension 4 or 8. This is no longer true in dimension 12; see Remark 3.3. In dimension 8, our proof requires a classical result on sums of four squares.

As far as condition (1.4) is concerned, it may be handled, for a fixed orientation, by using discriminants and local invariants of nondegenerate quadratic forms over $\mathbb{Q}$; see [11]. We give a similar interpretation for (1.4), where $\mathcal{A}$ is an arbitrary $\mathbb{Q}$–PDA, in Theorem 3.5, by analysing changes of orientation. For odd rank, the

answer depends only on local invariants. For even rank, both local invariants and discriminant are involved, in general; see Remark 3.6.

The results from Section 3 are applied in Section 4. Here, we construct 8–manifolds with interesting properties, starting from $WACI$'s which are homogeneous (that is, with $w_i = 2$, for all i). The integrality test from Theorem 3.5 is illustrated on two families of examples: one with odd rank (see Example 4.4), and the other with even rank (see Example 4.6). The even rank family has the remarkable property that the corresponding test, described in Theorem 3.5 (2), collapses to a single, simple, discriminant obstruction.

2. Signature and degree

The signature plays an important role in the smoothing problem described in § 1.2, via the Hirzebruch formula; see [9], [12], [2].

Let $\mathcal{A} = \mathbb{Q}[x_1, \dots, x_n]/(f_1, \dots, f_n)$ be an arbitrary $WACI$, as defined in §1.1. Among other things, S. Halperin showed in [5, Theorem 3] that $\mathcal{A}^*$ is a Poincaré duality algebra (1-connected and commutative), giving thus rise to $(\mathcal{A}, \cdot_\omega) \in W(\mathbb{Q})$, for any choice of orientation, $\omega \in \mathcal{A}^m \setminus \{0\}$. At the end of section §9 from [5], he raised the following question: is there an explicit way of computing the signature, $\sigma(\mathcal{A}, \cdot_\omega)$, in terms of the defining relations of $\mathcal{A}$?

Let us consider the associated C^∞ map germ,

$$f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0) \ , \tag{2.1}$$

having as components the defining polynomial relations of $\mathcal{A}$. Since $f^{-1}(0) = \{0\}$, the degree at 0 of f, $\deg(f)$, is defined, and may be computed in terms of regular values of f.

Using Theorem 1.2 from Eisenbud and Levine [4] (see also [1, p.103–104] and [6]), we may offer the following answer to the above signature problem.

Theorem 2.1. *Let $\mathcal{A}$ be an arbitrary $WACI$. Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be the C^∞ map germ associated to a system of defining relations for $\mathcal{A}$. Then:*

$$\sigma(\mathcal{A}, \cdot_\omega) = \deg(f) \ ,$$

for a good choice of orientation, ω.

Proof. In [4], the authors give a concrete way to algebraically compute the degree of a *finite* map. Namely let us consider $\mathcal{Q}(f) = \mathcal{C}_0^\infty(\mathbb{R}^n)/(f_1, \dots, f_n)$, where $\mathcal{C}_0^\infty(\mathbb{R}^n)$ is the ring of germs at 0 of smooth real-valued functions. A map $f = (f_1, \dots, f_n) : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ is called finite exactly when the corresponding $\mathcal{Q}(f)$ is a finite dimensional real vector space. Consequently 0 is isolated in $f^{-1}(0)$ and this allows one to consider the topological degree of f. In this set-up ([4, Theorem 1.2]) one can compute the degree of f as the signature of a symmetric bilinear form on $\mathcal{Q}(f)$. More precisely, if we define ω to be the class of the Jacobian $J = \det(\partial f_i/\partial x_j)$ in

$\mathcal{Q}(f)$, it follows from [4, Proposition 4.4(ii)] that $\omega \neq 0$. Pick any linear functional, $\varphi : \mathcal{Q}(f) \to \mathbb{R}$, such that $\varphi(\omega) > 0$. A symmetric bilinear form on $\mathcal{Q}(f)$ whose signature gives the degree is given by the following formula:

$$< p, q >_{\varphi} = \varphi(pq)\,, \quad \text{for} \quad p, q \in \mathcal{Q}(f)\,. \tag{2.2}$$

This result may be applied to the C^{∞} map germ (2.1), in the following way. Firstly, note that the natural algebra map

$$\Phi\colon \mathcal{A} \otimes \mathbb{R} = \mathbb{R}[x_1, \dots, x_n]/(f_1, \dots, f_n) \to \mathbb{R}[[x_1, \dots, x_n]]/(f_1, \dots, f_n) \tag{2.3}$$

is an isomorphism. Indeed, the surjectivity of Φ easily follows from the fact that $(x_1, \dots, x_n)^{m+1} \subset (f_1, \dots, f_n)$ (guaranteed by $\mathcal{A}^{>m} = 0$), while injectivity may be checked by a straightforward weight argument.

We may now use the isomorphism (2.3) to show that the natural algebra map

$$\Psi\colon \mathcal{A} \otimes \mathbb{R} \to \mathcal{Q}(f) \tag{2.4}$$

is an isomorphism as well (and consequently f is a finite map germ). To this end, consider the algebra map

$$\tau\colon \mathcal{C}_0^{\infty}(\mathbb{R}^n)/(f_1, \dots, f_n) \to \mathbb{R}[[x_1, \dots, x_n]]/(f_1, \dots, f_n)\,,$$

induced by Taylor series expansion. Since $\tau \circ \Psi = \Phi$, Ψ must be injective. The surjectivity of Ψ follows from the well-known fact (see e.g. [7, p.2]) that the ideal of k-flat functions (i.e., germs in $\mathcal{C}_0^{\infty}(\mathbb{R}^n)$ whose derivatives vanish at 0, up to order k) is contained in $(x_1, \dots, x_n)^{k+1}$, for any $k \geq 0$.

We infer from (1.3) and the isomorphism (2.4) that the class of J modulo $\mathcal{I}$, ω, belongs to $\mathcal{A}^m \setminus \{0\}$. This is the desired orientation. Indeed, we may use Ψ to identify the algebras $\mathcal{A} \otimes \mathbb{R}$ and $\mathcal{Q}(f)$, and then define φ to be the linear projection from the graded vector space $\mathcal{A}^* \otimes \mathbb{R}$ to $\mathcal{A}^m \otimes \mathbb{R} \equiv \mathbb{R}$. Since $\varphi(\omega) = 1$, the Eisenbud–Levine formula applies.

Obviously, the bilinear form (2.2) on $\mathcal{Q}(f)$ is identified with the Poincaré inner product on $\mathcal{A} \otimes \mathbb{R}$, $\cdot_{\omega}$. Our signature formula follows. □

3. Smoothing in small dimensions, and the integrality condition

3.1. **Small dimensions.** We begin by showing how the general smoothing problem becomes simpler, in dimensions up to 11.

Theorem 3.2. *Let $\mathcal{A}$ be a 1-connected Poincaré duality $\mathbb{Q}$-algebra with formal dimension m (not necessarily a WACI). Assume that $m \leq 11$. Then: $\mathcal{A}$ is smoothable (in the sense explained in §1.2) if and only if there is $\omega \in \mathcal{A}^{4k} \setminus \{0\}$ such that $(\mathcal{A}^{2k}, \cdot_{\omega}) \in W(\mathbb{Z})$, when $m = 4k$, and $\mathcal{A}$ is always smoothable, otherwise.*

Proof. For $m \not\equiv 0 \pmod 4$ arbitrary, smoothability follows from [12] and [2]. Assume then that $m = 4k$, with $k = 1$ or 2.

We have to show that $\mathcal{A}$ is smoothable, as soon as property (1.4) from §1.2 holds. Pick an orientation ω such that $\sigma := \sigma(\mathcal{A}^{2k}, \cdot_\omega) \geq 0$. We know that the Poincaré quadratic form on $\mathcal{A}^{2k}$ is a sum of t squares, $t \geq \sigma$, minus a sum of s squares.

If $k = 1$, then plainly $\mathcal{A}^* = H^*((\#_t \mathbb{CP}^2)\#(\#_s \overline{\mathbb{CP}^2}), \mathbb{Q})$, for $t + s > 0$, and $\mathcal{A}^* = H^*(S^4, \mathbb{Q})$, for $t + s = 0$.

If $k = 2$ and $\sigma = 0$, smoothability is guaranteed by (1.4); see §1.2.

Assume then that $k = 2$ and $\sigma > 0$. We claim that if the system

$$\begin{cases} a+b &= \sigma \\ 25a + 18b &= \sum_{i=1}^t \alpha_i^2 \end{cases} \tag{3.1}$$

has integer solutions, then $\mathcal{A}$ is smoothable.

Indeed, we may take the following algebraic Pontrjagin classes: $q_2 = (10a + 9b)\omega$, and $q_1 = \sum_{i=1}^t \alpha_i x_i$, where $\{x_i\}$ is the canonical basis of the positive definite part of $\mathcal{A}^4$. Set $N^8 = a \cdot \mathbb{CP}^4 + b \cdot \mathbb{CP}^2 \times \mathbb{CP}^2$. Using the second equation from (3.1), one may easily check that $\mathcal{A}$ and N have the same Pontrjagin numbers; see [9]. This implies, via the first equation from (3.1), that the Hirzebruch signature formula holds for $\mathcal{A}$, and we are done ([12], [2]).

We come back to the system (3.1). If $t \geq 4$, the theorem of Bachet de Méziriac–Lagrange [8, II.8] guarantees integer solutions.

In the remaining cases, σ must be 1, 2 or 3, and then (3.1) may be solved as follows:

$$\begin{cases} a = 1\,, \quad b = 0 \quad \text{and} \quad 25 = 5^2\,, & \text{for} \quad \sigma = 1\,; \\ a = 0\,, \quad b = 2 \quad \text{and} \quad 36 = 6^2\,, & \text{for} \quad \sigma = 2\,; \\ a = 1\,, \quad b = 2 \quad \text{and} \quad 61 = 5^2 + 6^2\,, & \text{for} \quad \sigma = 3\,. \end{cases}$$

This completes our proof. □

Remark 3.3. The range $m \leq 11$ is the best one for which the integrality condition alone guarantees smoothability. Indeed, $\mathcal{A} = \mathbb{Q}[x]/(x^3)$, with $\mid x \mid = 6$ has the integrality property, without being smoothable. See [10, §4.5].

3.4. **An integrality test.** In applications, we will need to check the integrality condition (1.4). This is clearly related to the theory of nondegenerate quadratic forms over $\mathbb{Q}$. We thus start by reviewing some relevant facts from [11].

To begin with, assume that $\mathcal{A}$ is an arbitrary $\mathbb{Q}$–PDA, with formal dimension $4k$. Set $r := \dim_{\mathbb{Q}} \mathcal{A}^{2k}$, and choose a $\mathbb{Q}$–basis of $\mathcal{A}^{2k}$. Pick any orientation, $\omega \in \mathcal{A}^{4k} \setminus \{0\}$, and denote by A_ω the matrix of $\cdot_\omega$. Note that $A_{\lambda\omega} = \lambda^{-1} \cdot A_\omega$, for any $\lambda \in \mathbb{Q}^*$. The condition $(\mathcal{A}^{2k}, \cdot_\omega) \in W(\mathbb{Z})$ translates to the fact that A_ω is equivalent over $\mathbb{Q}$ (in the classical sense, see [11, IV.1]) with a diagonal matrix of signs.

By a convenient choice of basis of $\mathcal{A}^{2k}$, we may suppose that $A_\omega = \mathrm{diag}(a_1, \dots, a_r)$. Then the discriminant of $\cdot_\omega$ is equal to $a_1 \cdots a_r$ (modulo $\mathbb{Q}^{*2}$). For each prime number p, one also has a *local invariant* at p, denoted by

$$\varepsilon_p(A_\omega) := \prod_{1 \le i < j \le r} (a_i, a_j)_p \in \{\pm 1\}\,,$$

where $(\cdot, \cdot)_p$ denotes the p–adic Hilbert symbol. From the classification theory ([11, IV.3]), we infer that

$$(\mathcal{A}^{2k}, \cdot_\omega) \in W(\mathbb{Z}) \Longleftrightarrow \mid a_1 \cdots a_r \mid \in \mathbb{Q}^{*2} \quad \text{and} \quad \varepsilon_p(A_\omega) = 1, \forall p \equiv 1(2)\,. \tag{3.2}$$

Our next result translates in similar terms condition (1.4), by taking into account changes of orientation.

Theorem 3.5. *Let $\mathcal{A}$ be an arbitrary $\mathbb{Q}$–PDA, of formal dimension $4k$. Then:*

(1) *Assume $r \equiv 1(2)$. For any orientation ω, there is $\lambda \in \mathbb{Q}^*$ such that the discriminant of $\cdot_{\lambda\omega}$ is 1 (modulo $\mathbb{Q}^{*2}$). Supposing that ω has the property that $a_1 \cdots a_r \in \mathbb{Q}^{*2}$, (1.4) is equivalent to*

$$\varepsilon_p(A_\omega) = 1,\ \forall\, p \equiv 1(2)\,.$$

(2) *Assume $r \equiv 0(2)$. Let ω be an arbitrary orientation. Set $\epsilon := \mathrm{sgn}(a_1 \cdots a_r)$. Two cases may occur:*

 (•) *$r \equiv 0(4)$ and $\epsilon = +1$, or $r \equiv 2(4)$ and $\epsilon = -1$. In this case, (1.4) is equivalent to*

$$\mid a_1 \cdots a_r \mid \in \mathbb{Q}^{*2} \quad \textit{and} \quad \varepsilon_p(A_\omega) = 1, \forall p \equiv 1(2)\,.$$

 (••) *$r \equiv 0(4)$ and $\epsilon = -1$, or $r \equiv 2(4)$ and $\epsilon = +1$. In this case, (1.4) is equivalent to*

$$\mid a_1 \cdots a_r \mid \in \mathbb{Q}^{*2} \quad \textit{and} \quad \varepsilon_p(A_\omega) = 1, \forall p \equiv 1(4)\,.$$

Proof. Part (1). The first assertion is easy: one may take for instance $\lambda = a_1 \cdots a_r$. As for the second one, it will follow from (3.2), as soon as the following claim is proved: if $(\mathcal{A}^{2k}, \cdot_{\lambda\omega}) \in W(\mathbb{Z})$, where $\lambda > 0$, then $(\mathcal{A}^{2k}, \cdot_\omega) \in W(\mathbb{Z})$. To verify this claim, note that the first condition in (3.2) implies that necessarily $\lambda \in \mathbb{Q}^{*2}$; this in turn ensures that $\varepsilon_p(\lambda^{-1} A_\omega) = \varepsilon_p(A_\omega)$, for all p (since Hilbert symbols are well-defined modulo squares, see [11, III.1]), and we are done.

Part (2). If r is even, it readily follows from (3.2) that (1.4) is equivalent to the fact that there is $\lambda \in \mathbb{Q}^*$, $\lambda > 0$, having the property that

$$\mid a_1 \cdots a_r \mid \in \mathbb{Q}^{*2} \quad \text{and} \quad \varepsilon_p(\lambda A_\omega) = 1, \forall p \equiv 1(2)\,. \tag{3.3}$$

It remains to compute the local invariants of $\lambda \cdot A_\omega$, at odd primes. This may be done as follows. Firstly, one may use the bilinearity of Hilbert symbols ([11, III.1]),

together with the first property from (3.3), to see that

$$\varepsilon_p(\lambda A_\omega) = \varepsilon_p(A_\omega) \cdot (\lambda, \lambda)_p^{\frac{r(r-1)}{2}} \cdot (\lambda, \epsilon)_p \,. \tag{3.4}$$

($\bullet$) In this case, elementary properties of Hilbert symbols ([11, III.1]) imply that (3.4) above reduces to

$$\varepsilon_p(\lambda A_\omega) = \varepsilon_p(A_\omega)\,, \tag{3.5}$$

and we are done.

($\bullet\bullet$) Similarly, in this case (3.4) becomes

$$\varepsilon_p(\lambda A_\omega) = \varepsilon_p(A_\omega) \cdot (\lambda, \lambda)_p \,. \tag{3.6}$$

Note that $(\mu\nu, \mu\nu)_p = (\mu, \mu)_p(\nu, \nu)_p$ and $(2, 2)_p = 1$ ([11, III.1]). It follows that we may assume in (3.3) that λ is a product of distinct odd primes, $\lambda = q_1 \cdots q_l$. Use [11, III.1] to compute

$$(\lambda, \lambda)_p = \prod_{i=1}^{l} (q_i, q_i)_p = \begin{cases} 1\,, & \text{for} \quad p \neq q_1, \dots, q_l\,; \\ (-1)^{\varepsilon(q_j)}\,, & \text{for} \quad p = q_j\,, \end{cases} \tag{3.7}$$

where $\varepsilon(q)$ denotes the residue class modulo 2 of $\frac{q-1}{2}$, as in [11]. We infer from (3.3), (3.6) and (3.7) that (1.4) implies the conditions from our statement.

Conversely, set

$$\{p = \text{ odd } \mid \varepsilon_p(A_\omega) = -1\} = \{q_1, \dots, q_l\}\,. \tag{3.8}$$

If all primes q_j appearing in (3.8) above are equal to 3 (modulo 4), then we may take $\lambda = q_1 \cdots q_l$, and (1.4) follows, again from (3.3), (3.6), and (3.7). Our proof is complete. $\square$

Remark 3.6. Let $(V, \cdot)$ be a symmetric inner product space over $\mathbb{Q}$ (alias, a nondegenerate quadratic $\mathbb{Q}$–form). For any k, $(V, \cdot)$ may obviously be realized as $(\mathcal{A}^{2k}, \cdot_\omega)$, where the oriented $\mathbb{Q}$–PDA $\mathcal{A}^*$ is $\mathbb{Q} \cdot 1$, in degree $* = 0$, V in degree $* = 2k$, $\mathbb{Q} \cdot \omega$ in degree $* = 4k$, and 0 otherwise, with product given by $\cdot$.

Note first that the condition $(\mathcal{A}^{2k}, \cdot_{\lambda\omega}) \in W(\mathbb{Z})$ may depend on $\lambda \in \mathbb{Q}^*$. For odd r, examples are easy to construct, using the discriminant obstruction from (3.2). For $r = 2$, for instance, a simple example is provided by the quadratic form with matrix $A = \mathrm{diag}(5, 5)$. Here, $(\mathcal{A}^{2k}, \cdot_\omega) \in W(\mathbb{Z})$, while $(\mathcal{A}^{2k}, \cdot_{3\omega}) \notin W(\mathbb{Z})$, even though the discriminant condition from (3.2) is verified.

Note also that, in general, the local invariants show up in an essential way, in our integrality test from Theorem 3.5 Part (2). Indeed, consider the matrix $A = \mathrm{diag}(1, 1, 1, 2, 5, 10)$. The associated $\mathbb{Q}$–PDA belongs to case ($\bullet\bullet$), and satisfies the discriminant condition therefrom. On the other hand, $\varepsilon_5(A) = -1$, as readily seen.

4. Homogeneous complete intersections

In this section, we want to apply Theorem 3.2 to $WACI$'s, in the nontrivial cases, that is, when the formal dimension is 4 or 8. We will restrict our attention to *homogeneous* $WACI$'s, i.e., those with $w_i = 2$ and $\mid f_i \mid = 2d_i \geq 4$, for all i. Both conditions are very natural. The first one simply means that each f_i is a homogeneous polynomial of degree d_i. The restrictions $d_i \geq 2$ $(1 \leq i \leq n)$ are imposed to avoid unnecessary redundancies, like $\mathbb{Q}[x]/(x) = \mathbb{Q}$.

It is straightforward to check that the formal dimension is equal to $4k$, with $k = 1$ or 2, precisely in the cases listed below (where $\underline{d}$ denotes $(d_1, \ldots, d_n)$, and $r := \dim_{\mathbb{Q}} \mathcal{A}^{2k}$); see (1.3).

Formal dimension 4:

$$\begin{array}{lll} (I_4) & \underline{d} = (2,2)\,; & r = 2\,. \\ (II_4) & \underline{d} = (3)\,; & r = 1\,. \end{array}$$

Formal dimension 8:

$$\begin{array}{lll} (I_8) & \underline{d} = (2,2,2,2)\,; & r = 6\,. \\ (II_8) & \underline{d} = (2,2,3)\,; & r = 4\,. \\ (III_8) & \underline{d} = (2,4)\,; & r = 2\,. \\ (IV_8) & \underline{d} = (3,3)\,; & r = 3\,. \\ (V_8) & \underline{d} = (5)\,; & r = 1\,. \end{array}$$

Note that, in the general homogeneous $WACI$ case, every degree vector, $\underline{d}$, may be realized by a smooth manifold. Indeed, $H^*(\prod_{i=1}^n \mathbb{CP}^{d_i-1}, \mathbb{Q}) = \otimes_{i=1}^n \mathbb{Q}[x_i]/(x_i^{d_i})$, with signature 1, when all d_i's are odd, and 0, otherwise. One may ask whether more interesting signatures may also arise from smooth manifolds. For instance, in case (I_8) above, the possible (non-negative) values of the signature are $0, 2, 4, 6$ (since $r = 6$). Our last main result completely clarifies this question.

Theorem 4.1. *All possible values of $\underline{d}$ and of the signature of homogeneous $WACI$'s with formal dimension $m = 4$ or 8 may be realized by smooth manifolds.*

The rest of this section will be devoted to the proof of the above theorem. The case $m = 4$ is easy.

Lemma 4.2. *Theorem 4.1 is true for $m = 4$.*

Proof. In case (I_4), $\mid \sigma \mid = 2$ or 0, realized by $H^*(\mathbb{CP}^2 \# \mathbb{CP}^2, \mathbb{Q}) = \mathbb{Q}[x_1, x_2]/(x_1^2 - x_2^2, x_1 x_2)$, and $H^*(\mathbb{CP}^1 \times \mathbb{CP}^1, \mathbb{Q}) = \mathbb{Q}[x_1, x_2]/(x_1^2, x_2^2)$ respectively.

In case (II_4), $\mid \sigma \mid = 1$, realized by $H^*(\mathbb{CP}^2, \mathbb{Q}) = \mathbb{Q}[x]/(x^3)$. $\square$

We move now to the case $m = 8$. In the next lemma, we take care of the subcases where the desired manifold may be obtained from known examples, by taking products and connected sums.

Lemma 4.3. *Theorem 4.1 is true for $m = 8$, in all cases different from (IV_8), $\sigma = 3$, and (I_8), $\sigma = 2$ or 6.*

Proof. If $\sigma = 0$ or 1, one may use products of complex projective spaces, as explained before.

Case (I_8), $\sigma = 4$: $H^*((\mathbb{CP}^2\#\mathbb{CP}^2) \times (\mathbb{CP}^2\#\mathbb{CP}^2), \mathbb{Q})$ is equal to

$$\mathbb{Q}[x_1, x_2, y_1, y_2]/(x_1^2 - x_2^2, x_1x_2, y_1^2 - y_2^2, y_1y_2)\,.$$

Case (II_8), $\sigma = 2$: $H^*((\mathbb{CP}^2\#\mathbb{CP}^2) \times \mathbb{CP}^2, \mathbb{Q}) = \mathbb{Q}[x_1, x_2, x_3]/(x_1^2 - x_2^2, x_1x_2, x_3^3)$.

Case (II_8), $\sigma = 4$: $\mathcal{A} = \mathbb{Q}[x_1, x_2, x_3]/(x_1^2 - x_3^2, x_2^2 - x_3^2, x_1x_2x_3)$ is a smoothable homogeneous $WACI$, with signature 4; see [10, Proposition 4.6].

Case (III_8), $\sigma = 2$: $H^*(\mathbb{CP}^4\#\mathbb{CP}^4, \mathbb{Q}) = \mathbb{Q}[x_1, x_2]/(x_1^4 - x_2^4, x_1x_2)$. □

To check the remaining cases, we will use the integrality test from Theorem 3.5. Case (IV_8) will follow from the analysis of the family below.

Example 4.4. Let $\mathcal{A}(c)$, $c \in \mathbb{Q}$, be the graded algebra

$$\mathbb{Q}[x, y]/(f_1 = x^3 - xy^2\,,\ f_2 = y^3 - cx^2y)\,,$$

with x and y of degree 2.

It is immediate to see that $\mathcal{A}(c)$ is a $WACI$ (homogeneous, belonging to case (IV_8)) precisely when $\{f_1 = f_2 = 0\} = \{0\}$ (over $\mathbb{C}$), that is, if and only if $c \neq 1$.

The next lemma completes the proof of Theorem 4.1, case (IV_8), and illustrates the arithmetic behind integrality condition (1.4).

Lemma 4.5. *Let $\{\mathcal{A}(c)\}_{c\neq 1}$ be the above $WACI$ family. Then:*

(1) *The absolute value of the signature of $\mathcal{A}(c)$ is $2 + \epsilon$, where $\epsilon = \mathrm{sgn}(c - 1)$.*
(2) *$\mathcal{A}(c)$ is smoothable $\Longleftrightarrow$ $\mid c - 1 \mid$ is a sum of two rational squares.*

Proof. Part (1). It is readily checked that the matrix of the Poincaré quadratic form on $\mathcal{A}^4(c)$, with respect to the basis $\{xy, x^2, x^2 - y^2\}$ and the orientation $\omega = (c-1)x^4$, is $A(c) = \mathrm{diag}(\frac{1}{c-1}, \frac{1}{c-1}, 1)$. Clearly, the signature of $\mathcal{A}(c)$ is as asserted.

Part (2). To decide the smoothability of $\mathcal{A}(c)$, we will use Theorem 3.5(1). Obviously, the orientation ω satisfies the required discriminant property. Therefore, $\mathcal{A}(c)$ is smoothable if and only if $(\frac{1}{c-1}, \frac{1}{c-1})_p = 1$, at all odd primes. This equivalent ([11, III.1]) with $(\frac{1}{\epsilon(c-1)}, \frac{1}{\epsilon(c-1)})_p = 1$, at all odd primes and also at ∞. By Hilbert's theorem (see [11, III.2]), this is further equivalent with $(\frac{1}{\epsilon(c-1)}, \frac{1}{\epsilon(c-1)})_p = 1$, at all primes and also at ∞.

The definition of Hilbert symbols ([11, III.1]) and the Hasse–Minkowski theorem ([11, IV.3]) together imply that this happens if and only if $\epsilon(c - 1)$ is a sum of two rational squares. The proof of Part (2) is complete. □

The last case of Theorem 4.1 ((I_8), $\sigma = 2$ or 6) will be covered by analysing a second family.

Example 4.6. Let us consider the family of graded algebras $\{\mathcal{B}(c)\}_{c\in\mathbb{Q}}$, with weight 2 generators, $\{x_i\}_{1\le i\le 4}$, and defining relations

$$\begin{cases} x_i^2 - x_4^2\,, & \text{for} \quad i \le 3\,, \\ \sum_{1\le i<j\le 4} x_i x_j - c x_4^2\,. \end{cases} \tag{4.1}$$

It is easy to see that (4.1) defines a $WACI$ (homogeneous, belonging to case (I_8)) if and only if $c \neq -2, 0, 6$. For $c = -1$, (4.1) defines the signature 6 algebra from [4, p.24] (which is not smoothable, by Lemma 4.7 below).

The next lemma completes the proof of Theorem 4.1. For its proof, we will resort to the integrality test from Theorem 3.5(2). At this point, it seems worthwhile pointing out that Part (2) of the lemma provides an interesting family of examples, where property (1.4) may be decided using only the (simple) discriminant obstruction. This simple behaviour cannot be expected, in general; see Remark 3.6.

Lemma 4.7. *Let $\{\mathcal{B}(c)\}_{c\neq -2,0,6}$ be the above $WACI$ family. Then:*

1. *The signature of $\mathcal{B}(c)$ is $0, \pm 2$ or ± 6.*
2. *$\mathcal{B}(c)$ is smoothable if and only if $\mid (c-6)(c+2) \mid \in \mathbb{Q}^{*2}$.*
3. *For $c = -3, 2$ and $-\frac{2}{5}$, the algebra $\mathcal{B}(c)$ is smoothable, with signature $0, 2$ and 6 respectively.*

Proof. Part (1). Our first task is to find a $\mathbb{Q}$–basis of $\mathcal{B}^4(c)$, and an orientation ω, with respect to which the matrix of the Poincaré inner product is diagonal. We will begin with the basis $\{x_i x_j\}_{1\le i<j\le 4}$. Set $y := x_i^2 \in \mathcal{B}^4(c)$, $1 \le i \le 4$. We claim that

$$x_1x_2x_3x_4 = \frac{c^2 - 4c - 6}{6} y^2\,, \tag{4.2}$$

and

$$y x_i x_j = \frac{c}{6} y^2\,, \quad \text{for} \quad 1 \le i < j \le 4\,, \tag{4.3}$$

so we may take $\omega = y^2$. Indeed, we infer from (4.1) that

$$c y x_1 x_2 = c y x_3 x_4 = y^2 + y(x_1 + x_2)(x_3 + x_4) + x_1x_2x_3x_4\,. \tag{4.4}$$

Adding all relations of type (4.4), we get (4.2). Using (4.2), (4.3) follows from (4.4).

Consider now the following basis in the middle dimension 4: $e_1 = x_1x_2 - x_3x_4, e_2 = x_1x_4 - x_3x_2, e_3 = x_1x_3 - x_2x_4, e_4 = x_1x_2 + x_3x_4, e_5 = x_1x_4 + x_3x_2, e_6 = x_1x_3 + x_2x_4$, and rescale the orientation to $\frac{(6-c)(c+2)}{3} y^2$.

With respect to these data, the intersection form is given by the following matrix:

$$B_1(c) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & b & b \\ 0 & 0 & 0 & b & a & b \\ 0 & 0 & 0 & b & b & a \end{pmatrix}$$

where $a = \frac{c(c-4)}{(6-c)(c+2)}, b = \frac{2c}{(6-c)(c+2)}$; use (4.2) and (4.3).

Firstly note that if $c = 4$ i.e. $a = 0$, the determinant of B_1 is positive, so clearly this case cannot produce $\sigma = 4$.

Let us now consider the case $c \neq 4$, i.e. $a \neq 0$.

Considering a new basis: $f_i = e_i, i \leq 4, f_5 = e_5 - e_6, f_6 = -2be_4 + ae_5 + ae_6$, we obtain a new matrix:

$$B_2(c) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(a-b) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2a(a-b)(a+2b) \end{pmatrix}$$

Rescaling our last basis to: $g_i = f_i, i \leq 5, g_6 = \frac{(6-c)(c+2)}{c^2} f_6$, we finally obtain the matrix:

$$B_3(c) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{c(c-4)}{(6-c)(c+2)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-2c}{c+2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{-2(c-4)}{c+2} \end{pmatrix} \tag{4.5}$$

Our assertion on signature from Part (1) easily follows by examining the distribution of signs on the diagonal of the matrix $B_3(c)$.

Part (3). Follows from Part (2).

Part (2). The algebra $\mathcal{B}(c)$ is smoothable if and only if it verifies the integrality test from Theorem 3.5 (2). The discriminant may be computed as $\det B_1(c) \equiv (6-c)(c+2)$ (modulo $\mathbb{Q}^{*2}$). This shows that we may assume from now on $c \neq 4$, and use the matrix $B_3(c)$. Set $\epsilon := \mathrm{sgn}((6-c)(c+2))$, and note that $\epsilon = +1$ (respectively $\epsilon = -1$) corresponds to the case (••) (respectively (•)). We have to show that, in both cases, the property $\mid (6-c)(c+2) \mid \in \mathbb{Q}^{*2}$ implies the restrictions

on the local invariants of $B := B_3(c)$ from Theorem 3.5. Set

$$\lambda_1 = \frac{-2c}{c+2}, \quad \lambda_2 = \frac{-2(c-4)}{c+2}, \quad \text{and} \quad \lambda_3 = \frac{c(c-4)}{(6-c)(c+2)}\,, \tag{4.6}$$

and note that $\lambda_3 \equiv \epsilon\lambda_1\lambda_2$ (modulo $\mathbb{Q}^{*2}$), by our assumption on the discriminant. By elementary manipulations with Hilbert symbols, we infer that

$$\varepsilon_p(B) = \begin{cases} (\lambda_1, \lambda_2)_p\,, & \text{if} \quad \epsilon = -1\,; \\ (\lambda_1, \lambda_2)_p \cdot (\lambda_1, \lambda_1)_p \cdot (\lambda_2, \lambda_2)_p\,, & \text{if} \quad \epsilon = +1\,. \end{cases} \tag{4.7}$$

The discriminant condition means that

$$\epsilon(6-c) = \frac{t^2}{s^2}(c+2)\,,$$

where t and s are relatively prime integers. Solve for c and substitute in (4.6), to obtain the following values (modulo $\mathbb{Q}^{*2}$) for $\lambda_{1,2}$:

$$\begin{cases} \lambda_1 = 2\epsilon t^2 - 6s^2\,, \\ \lambda_2 = 6\epsilon t^2 - 2s^2\,. \end{cases} \tag{4.8}$$

We are going to compute the Hilbert symbols appearing in (4.7), in terms of Legendre symbols; see [11, I.3 and Theorem III.1]. To do this, write

$$\begin{cases} \lambda_1 = p^{\alpha} u\,, \\ \lambda_2 = p^{\beta} v\,, \end{cases} \tag{4.9}$$

where $\alpha, \beta \in \mathbb{N}$, $u, v \in \mathbb{Z}$, and $u, v \not\equiv 0(p)$.

To finish our proof, we are going to show that $\varepsilon_p(B) = 1$, $\forall p \equiv 1(2)$ (when $\epsilon = -1$), and $\varepsilon_p(B) = 1$, $\forall p \equiv 1(4)$ (when $\epsilon = +1$).

Several cases may appear in (4.9). If $\alpha = \beta = 0$, then plainly $(\lambda_1, \lambda_2)_p = (\lambda_1, \lambda_1)_p = (\lambda_2, \lambda_2)_p = 1$, at all odd primes p. The case $\alpha, \beta > 0$ cannot occur, since this would imply (see (4.8)) that $s \equiv t \equiv 0(p)$. The remaining cases ($\alpha = 0, \beta > 0$ and $\alpha > 0, \beta = 0$) may be settled as follows.

For $\alpha = 0, \beta > 0$, one knows ([11, Theorem III.1]) that $(\lambda_1, \lambda_2)_p = \left(\frac{\lambda_1}{p}\right)^{\beta}$. Since $2s^2 \equiv 6\epsilon t^2\,(p)$, $\lambda_1 \equiv -16\epsilon t^2\,(p)$. Therefore, $(\lambda_1, \lambda_2)_p = (-\epsilon)^{\beta\varepsilon(p)}$. Similarly, for $\alpha > 0, \beta = 0$, one has $(\lambda_1, \lambda_2)_p = \left(\frac{\lambda_2}{p}\right)^{\alpha}$, with $\lambda_2 \equiv 16s^2\,(p)$, hence $(\lambda_1, \lambda_2)_p = 1$. By (4.7), this completes our proof, when $\epsilon = -1$.

Assume now $\epsilon = +1$. In this last case, we will also need $(\lambda_1, \lambda_1)_p = (-1)^{\alpha\varepsilon(p)}$, and $(\lambda_2, \lambda_2)_p = (-1)^{\beta\varepsilon(p)}$. When $p \equiv 1(4)$, both $(\lambda_1, \lambda_1)_p$ and $(\lambda_2, \lambda_2)_p$ are 1, which completes our proof (see (4.7)).

$\square$

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